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An extension to predicate logic of $\lambda\rho$-calculus

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Abstract

In [3], one of the authors introduced the system $\lambda\rho$-calculus in the case of implicational propositional logic. While the typed λ-calculus gives a natural deduction for intuitionistic logic, the typed $\lambda\rho$-calculus gives a natural deduction for classical logic. We extend it into predicate logic.

1 Typed $\lambda\rho$-calculus

Definition 1 (Individual terms).

Assume to have an infinite sequence of *individual variables* $u, v, w, \ldots$. *Individual terms* are defined as follows:

$$t \ ::= \ u \mid (ft \ldots t)$$

Individual ters are denoted by "s", "t".

Definition 2 (Types).

In types, we use three operators $\perp$, $\rightarrow$ and $\forall$. *Types* are defined as follows:

$$\tau \ ::= \ \perp \mid pt \ldots t \mid \tau \rightarrow \tau \mid \forall u.\tau$$

Types are denoted by lower-case Greek letters except "λ" and "ρ.

Definition 3 (Typed $\lambda\rho$-terms).

Assume to have an infinite sequence of λ-*variables* $x, y, z, w, \ldots$ and an infinite sequence of ρ-*variables* $a, b, c, d, \ldots$. *Typed $\lambda\rho$-terms* are defined as follows:

$$x^{\tau} : \tau \ \textit{(typed λ-variable)}, \qquad \frac{M : \sigma \rightarrow \tau \quad N : \sigma}{(MN) : \tau} \ (\textit{application}),$$

$$\frac{\begin{array}{c}[x^{\sigma} : \sigma] \\ \Pi \\ M : \tau\end{array}}{(\lambda x.M)^{\sigma\rightarrow\tau} : \sigma \rightarrow \tau} \ (\lambda\textit{-abstract}), \qquad \frac{\begin{array}{c}[a^{\tau} : \tau] \\ \Pi \\ M : \tau\end{array}}{(\rho a.M)^{\tau} : \tau} \ (\rho\textit{-abstract}),$$

$$\frac{a^{\tau} : \tau \quad M : \tau}{(a^{\tau}M)^{\sigma} : \sigma} \ (\rho\textit{-absurd}), \qquad \frac{M : \perp}{(\mathsf{A}M)^{\tau} : \tau} \ (\perp\textit{-absurd}),$$

$$\frac{M : \tau}{(\mathsf{J}M)_u : \forall u\tau}\ \textit{(generalization)}, \qquad \frac{M : \forall u\tau}{(\mathsf{F}M)_t : [t/u]\tau}\ \textit{(instantiation)}.$$

Typed $\lambda\rho$-terms are denoted by "M", "N", "P", "Q".

The type of a term M is denoted by $Type(M)$, and the set of types that a (λ- or ρ-) variable f has in M is denoted by $Type(f, M)$.

In *(λ-abstract)*, x is a λ-variable that satisfies $Type(x, M) \subseteq \{\sigma\}$. In *($\rho$-abstract)*, a is a ρ-variable that satisfies $Type(a, M) \subseteq \{\tau\}$. In *(generalization)*, for all of free variables in M, u has no free occurrence in the types that they have in M.

Note that ρ-variables are not terms.

We use the following notations:

- $f, g, \cdots$ denotes arbirary (λ- or ρ-) variables,
- $FV(M)$ denotes the set of free variables in M,
- $\lambda a.M$ denotes $\rho a.M$, so $\lambda ax.M \equiv \rho a.(\lambda x.M)$,

We identify α-equivalent terms.

Types on the shoulder of terms and parentheses are sometimes omitted from terms.

Example 4 (Peirce's Law).

$$\lambda xa.x^{(\alpha\to\beta)\to\alpha}(\lambda y.(a^\alpha y^\alpha)^\beta) : ((\alpha\to\beta)\to\alpha)\to\alpha.$$

This term is written in a tree form as follows:

$$\dfrac{\dfrac{\dfrac{x : (\alpha\to\beta)\to\alpha \qquad \dfrac{\dfrac{a^\alpha : \alpha \quad y^\alpha : \alpha}{(a^\alpha y^\alpha)^\beta : \beta}}{\alpha\to\beta}\ \lambda y}{\alpha}}{\alpha}\ \rho\alpha}{((\alpha\to\beta)\to\alpha)\to\alpha}\ \lambda x$$

To define the contraction of $\lambda\rho$-terms, we have to define several kinds of substitution. The following are easy to define.

- $[t/u]M$ the substitution of t for free occurrences of u in types on the structure of M,
- $[N/x]M$ the substitution of N for free occurrences of x in M where $Type(x, M) \subseteq \{Type(N)\}$,
- $[b/a]M$ the substitution of b for free occurrences of a in M,

Definition 5 (ρ-substitution).

For typed $\lambda\rho$-terms M, N and a ρ-variable a, we define $[\lambda x.b^\beta(x^{\alpha\to\beta}N)/a]M$ to be the result of substituting $\lambda x.b^\beta(x^{\alpha\to\beta}N)$ for every free occurrence of a in M, where $Type(a,M) \subseteq \{\alpha\to\beta\}$, $N:\alpha$, $x \notin FV(M)\cup FV(N)$, $b \notin FV(M)\cup FV(N)\cup\{a\}$.

Notice that the expression $\lambda x.b^\beta(x^{\alpha\to\beta}N)$ is not a typed $\lambda\rho$-term.

1. $[\lambda x.b(xN)/a]M \;\equiv\; M$ where $a \notin FV(M)$,
2. $[\lambda x.b(xN)/a](MQ) \;\equiv\; ([\lambda x.b(xN)/a]M\,[\lambda x.b(xN)/a]Q)$,
3. $[\lambda x.b(xN)/a]((\lambda y.M)^{\sigma\to\tau}) \;\equiv\; (\lambda z.[\lambda x.b(xN)/a][z^\sigma/y]M)^{\sigma\to\tau}$ where z is new,
4. $[\lambda x.b(xN)/a]((\rho c.M)^\tau) \;\equiv\; (\rho d.[\lambda x.b(xN)/a][d/c]M)^\tau$ where d is new,
5. $[\lambda x.b(xN)/a]((a^{\alpha\to\beta}M)^\sigma) \;\equiv\; (b^\beta([\lambda x.b(xN)/a]M\,N))^\sigma$,
6. $[\lambda x.b(xN)/a]((c^\tau M)^\sigma) \;\equiv\; (c^\tau\,[\lambda x.b(xN)/a]M)^\sigma$ where $c \not\equiv a$,
7. $[\lambda x.b(xN)/a]((\mathsf{A}M)^\sigma) \;\equiv\; (\mathsf{A}\,[\lambda x.b(xN)/a]M)^\sigma$,
8. $[\lambda x.b(xN)/a]((\mathsf{J}M)_u) \;\equiv\; (\mathsf{J}\,[\lambda x.b(xN)/a][v/u]M)_v$ where v is new,
9. $[\lambda x.b(xN)/a]((\mathsf{F}M)_t) \;\equiv\; (\mathsf{F}\,[\lambda x.b(xN)/a]M)_t$.

In 3, "z is new" means "z is a λ-varable that does not occur in the expression of the left side" *i.e.* z does not occur in M and N, $z \not\equiv x$, and $z \not\equiv y$. "d is new" in 4 and "v is new" in 8 are similar meanings respectively. We use the phrase "f/u *is new*" in a similar meaning after this.

Definition 6 (F_ρ-substitution).

For typed $\lambda\rho$-terms M and a ρ-variable a, we define $[\lambda x.b^{[t/u]\alpha}(\mathsf{F}x^{\forall u\alpha})_t/a]M$ to be the result of substituting $\lambda x.b^{[t/u]\alpha}(\mathsf{F}x^{\forall u\alpha})_t$ for every free occurrence of a in M, where $Type(a,M)\subseteq\{\forall u\alpha\}$, $x\notin FV(M)$, $b\notin FV(M)\cup\{a\}$.

Notice that the expression $\lambda x.b^{[t/u]\alpha}(\mathsf{F}x^{\forall u\alpha})_t$ is not a typed $\lambda\rho$-term.

1. $[\lambda x.b(\mathsf{F}x)/a]M \;\equiv\; M$ where $a \notin FV(M)$,
2. $[\lambda x.b(\mathsf{F}x)/a](MQ) \;\equiv\; ([\lambda x.b(\mathsf{F}x)/a]M\,[\lambda x.b(\mathsf{F}x)/a]Q)$,
3. $[\lambda x.b(\mathsf{F}x)/a]((\lambda y.M)^{\sigma\to\tau}) \;\equiv\; (\lambda z.[\lambda x.b(\mathsf{F}x)/a][z^\sigma/y]M)^{\sigma\to\tau}$ where z is new,
4. $[\lambda x.b(\mathsf{F}x)/a]((\rho c.M)^\tau) \;\equiv\; (\rho d.[\lambda x.b(\mathsf{F}x)/a][d/c]M)^\tau$ where d is new,
5. $[\lambda x.b(\mathsf{F}x)/a]((a^{\forall u\alpha}M)^\sigma) \;\equiv\; (b^{[t/u]\alpha}(\mathsf{F}\,[\lambda x.b(\mathsf{F}x)/a]M)_t)^\sigma$,
6. $[\lambda x.b(\mathsf{F}x)/a]((cM)^\sigma) \;\equiv\; (c\,[\lambda x.b(\mathsf{F}x)/a]M)^\sigma$ where $c \not\equiv a$,
7. $[\lambda x.b(\mathsf{F}x)/a]((\mathsf{A}M)^\sigma) \;\equiv\; (\mathsf{A}\,[\lambda x.b(\mathsf{F}x)/a]M)^\sigma$,
8. $[\lambda x.b(\mathsf{F}x)/a]((\mathsf{J}M)_v) \;\equiv\; (\mathsf{J}\,[\lambda x.b(\mathsf{F}x)/a][w/v]M)_w$ where w is new,
9. $[\lambda x.b(\mathsf{F}x)/a]((\mathsf{F}M)_s) \;\equiv\; (\mathsf{F}\,[\lambda x.b(\mathsf{F}x)/a]M)_s$.

Definition 7 (A_ρ-substitution).

For typed $\lambda\rho$-terms M and a ρ-variable a, we define $[\mathsf{A}/a]M$ to be the result of substituting A for every free occurrence of a in M, where $Type(a, M) \subseteq \{\perp\}$.

1. $[\mathsf{A}/a]M \;\equiv\; M \quad$ where $\;a \notin FV(M)$,
2. $[\mathsf{A}/a](MN) \;\equiv\; ([\mathsf{A}/a]M\,[\mathsf{A}/a]N)$,
3. $[\mathsf{A}/a]((\lambda x.M)^{\sigma\to\tau}) \;\equiv\; (\lambda x.[\mathsf{A}/a]M)^{\sigma\to\tau}$,
4. $[\mathsf{A}/a]((\rho b.M)^{\tau}) \;\equiv\; (\rho b.[\mathsf{A}/a]M)^{\tau}$,
5. $[\mathsf{A}/a]((a^{\perp}M)^{\sigma}) \;\equiv\; (\mathsf{A}\,[\mathsf{A}/a]M)^{\sigma}$,
6. $[\mathsf{A}/a]((c^{\tau}M)^{\sigma}) \;\equiv\; (c^{\tau}\,[\mathsf{A}/a]M)^{\sigma} \quad$ where $\;c \not\equiv a$,
7. $[\mathsf{A}/a]((\mathsf{A}(a^{\perp}M)^{\perp})^{\sigma}) \;\equiv\; (\mathsf{A}\,[\mathsf{A}/a]M)^{\sigma}$,
8. $[\mathsf{A}/a]((\mathsf{A}M)^{\sigma}) \;\equiv\; (\mathsf{A}\,[\mathsf{A}/a]M)^{\sigma}$,
9. $[\mathsf{A}/a]((\mathsf{J}M)_u) \;\equiv\; (\mathsf{J}\,[\mathsf{A}/a]M)_u$,
10. $[\mathsf{A}/a]((\mathsf{F}M)_t) \;\equiv\; (\mathsf{F}\,[\mathsf{A}/a]M)_t$.

Definition 8 ($\rho\beta$-contraction).

$$
\begin{aligned}
(\lambda x.M)^{\sigma\to\tau}N &\rhd_{1\beta} [N/x]M,\\
(\rho a.M)^{\sigma\to\tau}N &\rhd_{1\rho} (\rho b.([\lambda x.b^{\tau}(x^{\sigma\to\tau}N)/a]M)N)^{\tau},\\
& \qquad \textit{where } x,\ b \textit{ are new},\\
(a^{\alpha}M)^{\sigma\to\tau}N &\rhd_{1a} (a^{\alpha}M)^{\tau},\\
(\mathsf{A}M)^{\sigma\to\tau}N &\rhd_{1\mathsf{A}} (\mathsf{A}M)^{\tau},\\
(\mathsf{F}(\mathsf{J}M)_u)_t &\rhd_{1\mathsf{J}} [t/u]M,\\
(\mathsf{F}(\rho a.M)^{\forall u\tau})_t &\rhd_{1\mathsf{F}_\rho} (\rho b.(\mathsf{F}[\lambda x.b^{[t/u]\tau}(\mathsf{F}x^{\forall u\tau})_t/a]M)_t)^{[t/u]\tau}\\
& \qquad \textit{where } x,\ b \textit{ are new},\\
(\mathsf{F}(a^{\alpha}M)^{\forall u\tau})_t &\rhd_{1\mathsf{F}_a} (a^{\alpha}M)^{[t/u]\tau},\\
(\mathsf{F}(\mathsf{A}M)^{\forall u\tau})_t &\rhd_{1\mathsf{F}_\mathsf{A}} (\mathsf{A}M)^{[t/u]\tau},\\
(\mathsf{A}(\rho a.M)^{\perp})^{\tau} &\rhd_{1\mathsf{A}_\rho} (\mathsf{A}\,[\mathsf{A}/a]M)^{\tau},\\
(\mathsf{A}(a^{\alpha}M)^{\perp})^{\tau} &\rhd_{1\mathsf{A}_a} (a^{\alpha}M)^{\tau}.
\end{aligned}
$$

Example 9 (ρ-contraction).

$$(\rho a.(ay))N \;\;\rhd_{1\rho}\;\; \rho b.([\lambda x.b(xN)/a](ay))N \equiv \rho b.(b(yN))N$$

These therms before and after the contraction are written in tree forms as follows:

$$\dfrac{\dfrac{\dfrac{a:\sigma\to\tau \quad y:\sigma\to\tau}{(ay)^{\sigma\to\tau}:\sigma\to\tau}}{\sigma\to\tau}\,\rho a \qquad \begin{matrix}\Pi\\ N:\sigma\end{matrix}}{(\rho a.(ay))N:\tau} \quad \rhd_{1\rho} \quad \dfrac{\dfrac{b:\tau \quad \dfrac{y:\sigma\to\tau \quad \begin{matrix}\Pi\\ N:\sigma\end{matrix}}{\tau}}{(b(yN))^{\sigma\to\tau}:\sigma\to\tau} \qquad \begin{matrix}\Pi\\ N:\sigma\end{matrix}}{\rho b.(b(yN))N:\tau}\,\rho b$$

Definition 10 ($\rho\beta$-contraction, $\rho\beta$-reduction).

A "$\rho\beta$-redex" is any typed $\lambda\rho$-term of form $((\lambda x.M)^{\sigma\to\tau}N)$, $((\rho a.M)^{\sigma\to\tau}N)$, $\ldots$, or $(\mathsf{A}(a^{\alpha}M)^{\perp})^{\tau}$.

If M contains a $\rho\beta$-redex $\underline{P}$ and N is the result of replacing $\underline{P}$ by its contractum, we say "M $\rho\beta$-contracts to N", or $M \rhd_{1\rho\beta} N$.

If $M \rhd_{1\rho\beta} M_1 \rhd_{1\rho\beta} M_2 \rhd_{1\rho\beta} \cdots \rhd_{1\rho\beta} M_n \equiv N$ $(n \geq 0)$, we say "M $\rho\beta$-reduces to N", or $M \rhd_{\rho\beta} N$.

2 Subject-reduction theorem

Lemma 11.

For any typed $\lambda\rho$-terms M, N,

- $\mathit{Type}([t/u]M) = [t/u]\mathit{Type}(M)$,
- $\mathit{Type}([N/x]M) = \mathit{Type}(M)$ and $\mathit{FV}([N/x]M) \subseteq (\mathit{FV}(M) - \{x\}) \cup \mathit{FV}(N)$,
- $\mathit{Type}([\lambda x.b^{\beta}(x^{\alpha\to\beta}N)/a]M) = \mathit{Type}(M)$ and $\mathit{FV}([\lambda x.b^{\beta}(x^{\alpha\to\beta}N)/a]M) \subseteq (\mathit{FV}(M) - \{a\}) \cup \mathit{FV}(N)$,
- $\mathit{Type}([\lambda x.b^{[t/u]\tau}(\mathsf{F}x^{\forall u\tau})_t/a]M) = \mathit{Type}(M)$ and $\mathit{FV}([\lambda x.b^{[t/u]\tau}(\mathsf{F}x^{\forall u\tau})_t/a]M) \subseteq (\mathit{FV}(M) - \{a\}) \cup \{b\}$,
- $\mathit{Type}([\mathsf{A}/a]M) = \mathit{Type}(M)$ and $\mathit{FV}([\mathsf{A}/a]M) \subseteq \mathit{FV}(M) - \{a\}$.

Proof. By induction on the structure of M. □

Theorem 12 (Subject-reduction theorem).

For any typed $\lambda\rho$-terms M, N,

$$M \rhd_{\rho\beta} N \;\Rightarrow\; \mathit{Type}(N) = \mathit{Type}(M) \text{ and } \mathit{FV}(N) \subseteq \mathit{FV}(M).$$

Proof. It is enough to take care of the case in which M is a redex and N is its contractum. By the previous lemmas, it is easy to prove. □

3 Church-Rosser theorem

Theorem 13 (Strong normalization theorem).
For any typed $\lambda\rho$-term M, all $\rho\beta$-reductions starting at M are finite.

Proof. Similar to the case of propositional logic. cf. [3]. □

Theorem 14 (Theorem 3.10 in [2]).
If a translation † has the following properties, then $\rhd_{\rho\beta}$ has a Church-Rosser property.

For any typed $\lambda\rho$-terms M, N,

$$\begin{array}{lll} \langle 1 \rangle & M \rhd_{\rho\beta} M^{\dagger}, & \\ \langle 2 \rangle & M \rhd_{1\rho\beta} N & \Rightarrow \quad N \rhd_{\rho\beta} M^{\dagger}, \\ \langle 3 \rangle & M \rhd_{1\rho\beta} N & \Rightarrow \quad M^{\dagger} \rhd_{\rho\beta} N^{\dagger}. \end{array}$$

Lemma 15.
With the strong normalization theorem of $\lambda\rho$-terms, if a translation † has the following properties, then $\rhd_{\rho\beta}$ has a Church-Rosser property.

For any typed $\lambda\rho$-terms M, N,

$$\begin{array}{lll} \langle 1 \rangle & M \rhd_{\rho\beta} M^{\dagger}, & \\ \langle 2 \rangle & M \rhd_{1\rho\beta} N & \Rightarrow \quad N \rhd_{\rho\beta} M^{\dagger}, \end{array}$$

Proof. It is enough to prove that normal form is decided uniquely on the assumption. cf. [2]. □

Definition 16 (Translation †).

1. $(x^{\tau})^{\dagger} \equiv x^{\tau}$,
2. $((\lambda x.M)^{\sigma\to\tau}N)^{\dagger} \equiv [N^{\dagger}/x]M^{\dagger}$,
3. $((\rho a.M)^{\sigma\to\tau}N)^{\dagger} \equiv (\rho b.([\lambda x.b^{\tau}(x^{\sigma\to\tau}N^{\dagger})/a]M^{\dagger})N^{\dagger})^{\tau}$,
4. $((a^{\alpha}M)^{\sigma\to\tau}N)^{\dagger} \equiv (a^{\alpha}M^{\dagger})^{\tau}$,
5. $((\mathsf{A}M)^{\sigma\to\tau}N)^{\dagger} \equiv (\mathsf{A}M^{\dagger})^{\tau}$,
6. $((\mathsf{F}(\mathsf{J}M)_u)_t)^{\dagger} \equiv [t/u]M^{\dagger}$,
7. $((\mathsf{F}(\rho a.M)^{\forall u\tau})_t)^{\dagger} \equiv (\rho b.(\mathsf{F}\,[\lambda x.b^{[t/u]\tau}(\mathsf{F}x^{\forall u\tau})_t/a]M^{\dagger})_t)^{[t/u]\tau}$,
8. $((\mathsf{F}(a^{\alpha}M)^{\forall u\tau})_t)^{\dagger} \equiv (a^{\alpha}M^{\dagger})^{[t/u]\tau}$,
9. $((\mathsf{F}(\mathsf{A}M)^{\forall u\tau})_t)^{\dagger} \equiv (\mathsf{A}M^{\dagger})^{[t/u]\tau}$,
10. $((\mathsf{A}(\rho a.M)^{\perp})^{\tau})^{\dagger} \equiv (\mathsf{A}\,[\mathsf{A}/a]M^{\dagger})^{\tau}$,
11. $((\mathsf{A}(a^{\alpha}M)^{\perp})^{\tau})^{\dagger} \equiv (a^{\alpha}M^{\dagger})^{\tau}$,
12. $(MN)^{\dagger} \equiv M^{\dagger}N^{\dagger}$,

13. $((\lambda x.M)^{\sigma\to\tau})^\dagger \equiv (\lambda x.M^\dagger)^{\sigma\to\tau}$,

14. $((\rho a.M)^{\tau})^\dagger \equiv (\rho a.M^\dagger)^{\tau}$,

15. $((a^\alpha M)^{\sigma})^\dagger \equiv (a^\alpha M^\dagger)^{\sigma}$,

16. $((\mathsf{A}M)^{\sigma})^\dagger \equiv (\mathsf{A}M^\dagger)^{\sigma}$,

17. $((\mathsf{J}M)_u)^\dagger \equiv (\mathsf{J}M^\dagger)_u$,

18. $((\mathsf{F}M)_t)^\dagger \equiv (\mathsf{F}M^\dagger)_t$.

Here we choose to apply the rule with smallest number if many rules can apply to M.

Lemma 17.
For any typed $\lambda\rho$-term M, N, if $M \rhd_{\rho\beta} N$ then

- $[t/u]M \rhd_{\rho\beta} [t/u]N$,
- $[Q/x]M \rhd_{\rho\beta} [Q/x]N$,
- $[M/x]Q \rhd_{\rho\beta} [N/x]Q$,
- $[b/a]M \rhd_{\rho\beta} [b/a]N$,
- $[\lambda x.b^\beta(x^{\alpha\to\beta}Q)/a]M \rhd_{\rho\beta} [\lambda x.b^\beta(x^{\alpha\to\beta}Q)/a]N$,
- $[\lambda x.b^\beta(x^{\alpha\to\beta}M)/a]Q \rhd_{\rho\beta} [\lambda x.b^\beta(x^{\alpha\to\beta}N)/a]Q$,
- $[\lambda x.b^{[t/u]\alpha}(\mathsf{F}x^{\forall u\alpha})_t/a]M \rhd_{\rho\beta} [\lambda x.b^{[t/u]\alpha}(\mathsf{F}x^{\forall u\alpha})_t/a]N$,
- $[\mathsf{A}/a]M \rhd_{\rho\beta} [\mathsf{A}/a]N$.

Lemma 18. *For all $\lambda\rho$-term M,*

$$FV(M^\dagger) \subseteq FV(M).$$

Proof. By induction on the structure of M. □

Lemma 19. *For all $\lambda\rho$-term M,*

$$M \rhd_{\rho\beta} M^\dagger.$$

Proof. By induction on the structure of M. □

Lemma 20. *For all $\lambda\rho$-term M, N,*

$$M \rhd_{1\rho\beta} N \quad\Rightarrow\quad N \rhd_{\rho\beta} M^\dagger.$$

Proof. By induction on the structure of M. □

Theorem 21 (Church-Rosser theorem).
For any typed $\lambda\rho$-terms M, P, Q, if $M \rhd_{\rho\beta} P$ and $M \rhd_{\rho\beta} Q$, then there exists a typed $\lambda\rho$-term N such that

$$P \rhd_{\rho\beta} N \quad\text{and}\quad Q \rhd_{\rho\beta} N.$$

4 Subformula property

Definition 22 (Subterm).

1. $Subt(x^\tau) \;=\; \{x^\tau\}$,
2. $Subt((MN)) \;=\; Subt(M) \cup Subt(N) \cup \{(MN)\}$,
3. $Subt((\lambda x.M)^{\sigma\to\tau}) \;=\; Subt(M) \cup \{x^\sigma\} \cup \{(\lambda x.M)^{\sigma\to\tau}\}$,
4. $Subt((\rho a.M)^\tau) \;=\; Subt(M) \cup \{a^\tau\} \cup \{(\rho a.M)^\tau\}$,
5. $Subt((a^\tau M)^\sigma) \;=\; Subt(M) \cup \{a^\tau\} \cup \{(a^\tau M)^\sigma\}$,
6. $Subt((\mathsf{A}M)^\sigma) \;=\; Subt(M) \cup \{(\mathsf{A}M)^\sigma\}$,
7. $Subt((\mathsf{J}M)_u) \;=\; Subt(M) \cup \{(\mathsf{J}M)_u\}$,
8. $Subt((\mathsf{F}M)_t) \;=\; Subt(M) \cup \{(\mathsf{F}M)_t\}$.

Definition 23 (Subfomula).
For any types α, β, "*α is a subformula of β*" or $\alpha \leq \beta$ is defined inductively as follows:

$$\begin{array}{l} \delta \;\leq\; \delta, \\ \delta \;\leq\; \alpha \;\Rightarrow\; \delta \;\leq\; \alpha \to \beta \text{ and } \delta \;\leq\; \beta \to \alpha, \\ \delta \;\leq\; [t/u]\alpha \;\Rightarrow\; \delta \;\leq\; \forall u\alpha. \end{array}$$

Theorem 24 (Subformula property).
For any typed $\lambda\rho$-term M, if M is a $\rho\beta$-normal form then for any type δ

$$\delta \in Type(Subt(M)) \;\;\Rightarrow\;\; \delta \leq Type(FV(M) \cup \{M\}).$$

Proof. By induction on the structure of M. □

5 Correspondence to Gentzen's LK

Theorem 25 (LK to HK).
For any set of types Γ and a type γ, if a sequent $\Gamma \Rightarrow \gamma$ is provable in LK, then $\Gamma \vdash_{HK} \gamma$.

Lemma 26 (HK to $\lambda\rho$-terms).
For any set of types Γ and a type γ, if $\Gamma \vdash_{HK} \gamma$, then there exists a typed $\lambda\rho$-term M such that $\Gamma \supseteq Type(FV_\lambda(M))$, $Type(FV_\rho(M)) = \phi$, $Type(M) = \gamma$.

Proof. By induction on the proof of $\Gamma \vdash_{HK} \gamma$. □

Lemma 27.
For any typed $\lambda\rho$-term M, if M is a $\rho\beta$-normal form then a sequent

$$Type(FV_\lambda(M)) \Rightarrow Type(FV_\rho(M) \cup \{M\})$$

is provable in LK without cut.

Proof. By induction on the structure of M. $\square$

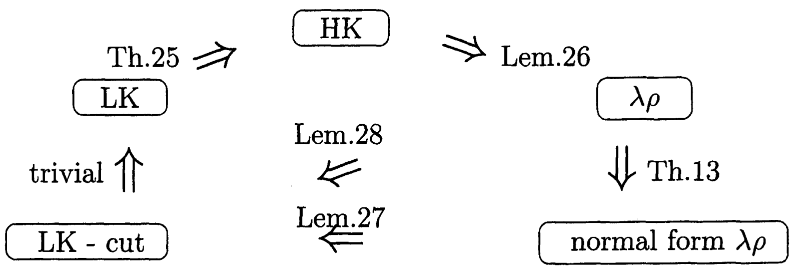
Lemma 28 ($\lambda\rho$-terms to LK).
For any typed $\lambda\rho$-term M, a sequent

$$Type(FV_\lambda(M)) \Rightarrow Type(FV_\rho(M) \cup \{M\})$$

is provable in LK without cut.

Proof. By the strong normalization theorem of $\lambda\rho$-terms and the previous lemma. $\square$

The previous lemmas are written in a figure as follows:



Theorem 29.
For any set of types Γ and Θ, a sequent $\Gamma \Rightarrow \Theta$ is provable in LK if and only if there exists a typed $\lambda\rho$-term M such that $\Gamma \supseteq Type(FV_\lambda(M))$ and $\Theta \supseteq Type(FV_\rho(M) \cup \{M\})$.

Proof. By the previous lemmas. $\square$

Theorem 30 (Cut elimination theorem of LK).
For any set of types Γ and Θ, if a sequent $\Gamma \Rightarrow \Theta$ is provable in LK, then it is also provable in LK without cut.

Proof. By the previous lemmas. $\square$

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